

A Canonical Version for Partition Regular Systems of Linear Equations

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We prove a canonical (unrestricted) version of Deuber's partition theorem for (m, p, c) -sets (*Math. Z.* **133** (1973), 109–123). This implies a canonical result for partition regular systems of linear equations studied by Rado (*Math. Z.* **36** (1933), 424–480). This is a common generalization of former results of Erdős and Graham (*Enseign. Math.* **28** (1980)) concerning arithmetic progressions and of Prömel and Voigt (*J. Combin. Theory Ser. A* **35**, 309–327) concerning the so-called Rado–Folkman–Sanders theorem on finite sums. © 1986 Academic Press, Inc.

1. INTRODUCTION

Let $\mathbb{N} = \{1, 2, \dots\}$ denote the set of natural numbers, \mathbb{Z} the set of integers and \mathbb{Q} the set of rationals. For integers $a \leq b$ let $[a, b]$ denote the interval $\{z \in \mathbb{Z} | a \leq z \leq b\}$.

Van der Waerden proved the following result on arithmetic progressions:

1.1. THEOREM [6]. *For every pair of natural numbers (δ, k) there exists a natural number n which satisfies:*

for every coloring $\Delta: [1, n] \rightarrow [1, \delta]$ there is in $[1, n]$ a $(k+1)$ -term arithmetic progression $a, a+d, \dots, a+kd$ such that the restriction of Δ to $\{a, a+d, \dots, a+kd\}$ is a constant mapping.

Van der Waerden considered colorings with only a finite number of colors. If one considers arbitrary colorings of the natural numbers, one cannot expect to obtain a monochromatic arithmetic progression. However, Erdős and Graham proved that one can restrict to two types of colorings.

1.2. THEOREM [2]. *For every natural number k there exists a natural number n with the following property:*

for every coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ there exists in $[1, n]$ a $(k+1)$ -term arithmetic progression $a, a+d, \dots, a+kd$ such that $\Delta \upharpoonright \{a, a+d, \dots, a+kd\}$ is either a constant or one-to-one coloring. (*)

For a given natural number the least natural number n , which satisfies (*), will be denoted by $EG(k)$. Theorem 1.2 is often called the “canonical version” of Theorem 1.1. Rado generalized van der Waerden’s theorem to systems of linear equations. For this he needed:

1.3. DEFINITION. A system of linear equations $A\mathbf{x} = \mathbf{b}$ with $A \in \mathbb{Q}^{v \times w}$ and $\mathbf{b} \in \mathbb{Q}^{v \times 1}$ is partition regular iff for every coloring $\Delta: \mathbb{N} \rightarrow [1, \delta]$ of the natural numbers with finitely many colors there exists a solution $(\alpha_1, \dots, \alpha_w)' \in \mathbb{N}^w$ of $A\mathbf{x} = \mathbf{b}$ such that $\Delta \upharpoonright \{\alpha_1, \dots, \alpha_w\}$ is a constant mapping.

By linearity it is only necessary to consider matrices with integral coefficients. Moreover, we restrict to matrices with a finite number of columns and rows.

1.4. DEFINITION. A matrix $A = (\mathbf{a}^1, \dots, \mathbf{a}^w)$ with integral coefficients has column property iff there exists a partition of the set of column indices of A , i.e., $[1, w] = I_0 \cup \dots \cup I_m$, such that the following are valid:

- (i) $\sum_{i \in I_0} \mathbf{a}^i = \mathbf{0}$, and if $m \geq 1$,
- (ii) for every $j = 0, \dots, m-1$ there are rational numbers s_{1j}, \dots, s_{vj} such that

$$\sum_{i \in I_0 \cup \dots \cup I_j} \mathbf{a}^i s_{ij} + \sum_{i \in I_{j+1}} \mathbf{a}^i = \mathbf{0}.$$

Rado proved the following result, which generalizes van der Waerden’s theorem:

1.5. THEOREM [4]. For a matrix A with integral coefficients the following statements are equivalent:

- (i) $A\mathbf{x} = \mathbf{0}$ is partition regular;
- (ii) A has column property.

For inhomogeneous systems of linear equations the following is valid:

1.6. THEOREM [4]. Let $A \in \mathbb{Z}^{v \times w}$ and $\mathbf{b} \in \mathbb{Z}^{v \times 1}$. Then $A\mathbf{x} = \mathbf{b}$ is partition regular iff one of the following statements is true:

- (i) there exists $a \in \mathbb{N}$ such that $A(a, \dots, a)^t = \mathbf{b}$
- (ii) there exists $a \in \mathbb{Z} \setminus \mathbb{N}$ such that $A(a, \dots, a)^t = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$ is partition regular.

Deuber [1] extended Rado's results; the crucial step was the introduction of (m, p, c) -sets.

1.7 DEFINITION. Let m be a nonnegative integer and p, c, x_0, \dots, x_m be natural numbers. Let $M_i^{p,c}(x_i, \dots, x_m) = \{cx_i + \sum_{j=i+1}^m \alpha_j x_j \mid \alpha_j \in [-p, p]\}$. A subset M of the natural numbers is an (m, p, c) -set iff there exist natural numbers x_0, \dots, x_m such that $M = M^{p,c}(x_0, \dots, x_m) = \cup_{i=0}^m M_i^{p,c}(x_i, \dots, x_m)$.

In this paper we shall make use of [1] and give "canonical versions" of these theorems. This means that we shall consider arbitrary colorings of the natural numbers and show how the solutions behave with respect to these colorings. It will turn out that one cannot restrict to two cases, but three will suffice. In particular, we get as a corollary a result of Prömel and Voigt [3]—a "canonical version" of the so called Rado–Folkman–Sanders theorem.

2. HOMOGENEOUS EQUATIONS

Since the situation for a single linear equation turns out to be different from the general case of arbitrary systems of linear equations, we consider it first.

Let a_1, \dots, a_w be nonzero integers with $a_1 + \dots + a_w \neq 0$. If $a_1 x_1 + \dots + a_w x_w = 0$ has a solution in the natural numbers, then there exists a solution $\gamma_1, \dots, \gamma_w$ with $\{\gamma_1, \dots, \gamma_w\} = \{\gamma, \gamma'\}$, where $\gamma, \gamma' \in \mathbb{N}$. For an arbitrary coloring $\Delta: \mathbb{N} \rightarrow \mathbb{N}$ the restriction of Δ to $\{\gamma, \gamma'\}$ is either constant or one-to-one. Thus the canonical cases for a single homogeneous linear equation are the constant and the injective coloring.

Remark. Note that " $\Delta \upharpoonright \{\gamma_1, \dots, \gamma_w\}$ one-to-one" does not mean $\Delta(\gamma_i) \neq \Delta(\gamma_j)$ whenever $i \neq j$ but rather that different numbers γ_i and γ_j have different colors.

For the general case of homogeneous partition regular systems of linear equations we need the following "canonical version" of a theorem of Deuber [1] on (m, p, c) -sets:

2.1 THEOREM. *Let m, p, c be natural numbers. Then there exists a natural number n which satisfies:*

for every coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ there is an (m, p, c) -set $M = \cup_{i=0}^m M_i^{p,c}(x_i, \dots, x_m)$ in $[1, n]$ with $M_i^{p,c}(x_i, \dots, x_m) \cap$

$M_j^{p,c}(x_j, \dots, x_m) = \emptyset$ for every $i < j$ in $[0, m]$ such that one of the following cases occurs:

- (i) $\Delta \upharpoonright M$ is constant;
- (ii) $\Delta \upharpoonright M$ is one-to-one;
- (iii) $\Delta \upharpoonright M_i^{p,c}(x_i, \dots, x_m)$ is constant for every $i = 0, \dots, m$ and $\Delta \upharpoonright \{cx_0, \dots, cx_m\}$ is one-to-one. (*)

The number of these cases is minimal.

For given natural numbers m, p, c we abbreviate statement (*) of Theorem 2.1 as usual to $n \rightarrow (m, p, c)_{can}$.

Proof. It will be shown first that all three cases can occur. Obviously, cases (i) and (ii) cannot be omitted, since the natural numbers can be colored constantly or injectively. However, these do not suffice. Since every natural number x has a representation $x = 3^{x'}x''$ with $x' \in \mathbb{N} \cup \{0\}$, $x'' \in \mathbb{N}$ and $x'' \not\equiv 0 \pmod{3}$, consider the coloring $\Delta: \mathbb{N} \rightarrow \mathbb{N}$ with $\Delta(x) = x' + 1$.

Let $M^{p,c}(x_0, \dots, x_m)$ be an (m, p, c) -set and let c, x_0, \dots, x_m have the representation $c = 3^{c'} \cdot c''$, $x_0 = 3^{x'_0} \cdot x''_0, \dots, x_m = 3^{x'_m} \cdot x''_m$.

The following three cases can occur:

- (α) $c' + x'_0 < x'_1$ —then $\Delta(cx_0) = \Delta(cx_0 - x_1) = \Delta(cx_0 + x_1) \neq \Delta(cx_1)$;
- (β) $c' + x'_0 = x'_1$ —so exactly two elements of the set $\{cx_0, cx_0 - x_1, cx_0 + x_1\}$ are colored the same;
- (γ) $c' + x'_0 > x'_1$ —then $\Delta(cx_0 - x_1) = \Delta(cx_0 + x_1) \neq \Delta(cx_0)$.

Thus cases, (i), (ii) do not occur.

Now it will be shown that three cases are sufficient. Let m, p, c be natural numbers. Define recursively nonnegative integers n_i, n'_i for $i = 1, \dots, m^2 + m + 1$: let $n_{m^2+m+1} = 0$ and $n'_{m^2+m+1} = 1$ and for $i = 1, \dots, m^2 + m$ let n_i, n'_i be defined by

$$\left[\frac{n_i}{1 + ((2p+1)^{m_i} - 1)/2p} \right] = 2A_i + 1$$

with $m_i = \min\{m^2 + m + 1 - i, m\}$ and $A_i = pc^{m^2+m-i} \cdot \prod_{j=i+1}^{m^2+m+1} n'_j$ and $n'_i = EG(n_i)$.

Let $\Delta: [1, cn'_1] \rightarrow \mathbb{N}$ be a coloring. Construct recursively $(m^2 + m + 1)$ -many arithmetic progressions each colored either constantly or injectively with respect to Δ as follows.

Let $x_0 = 1$. Consider the set $\{cx_0, 2cx_0, \dots, n'_1 cx_0\}$; Theorem 1.2 together with the definition of n'_1 provides an $(n_1 + 1)$ -term arithmetic progression $cx'_0, cx'_0 + x_1, \dots, cx'_0 + n_1 x_1$ colored either constantly or injectively with respect to Δ .

Let i -many arithmetic progressions $cx'_j, cx'_j + x_{j+1}, \dots, cx'_j + n_{j+1}x_{j+1}$ for $j=0, \dots, i-1$ be constructed, each colored constantly or injectively with respect to Δ , were $\{cx'_j, cx'_j + x_{j+1}, \dots, cx'_j + n_{j+1}x_{j+1}\} \subseteq \{cx_j, 2cx_j, \dots, n'_{j+1}cx_j\}$. Consider the set $\{cx_i, 2cx_i, \dots, n'_{i+1}cx_i\}$; by Theorem 1.2 and the choice of n'_{i+1} there exists an $(n_{i+1}+1)$ -term arithmetic progression $cx'_i, cx'_i + x_{i+1}, \dots, cx'_i + n_{i+1}x_{i+1}$ colored either constantly or injectively with respect to Δ .

Finally obtain arithmetic progressions $M_i = \{cx'_i + \alpha x_{i+1} | \alpha \in [0, n_{i+1}]\}$, $i=0, 1, \dots, m^2+m$, with the following properties:

$$\left. \begin{array}{l} \Delta \upharpoonright M_i \text{ is either constant or one-to-one and} \\ M_i \subseteq \{cx_i, 2cx_i, \dots, n'_{i+1}cx_i\} \text{ for every } i \in [0, m^2+m] \end{array} \right\}. \quad (*)$$

The pigeon-hole principle applied to $[0, m^2+m]$ provides one of the $(m+1)$ -element subsets X_0, X_1, X_2 which induce the following properties:

- $\Delta \upharpoonright \cup_{x \in X_0} M_x$ is constant;
- $\Delta \upharpoonright M_x$ is one-to-one for every $x \in X_1$;
- $\Delta \upharpoonright M_x$ is constant for every $x \in X_2$ and $\Delta(M_{x_1}) \neq \Delta(M_{x_2})$ for every $x_1 \neq x_2$ in X_2 .

Concentrate on X_1 ; X_0 and X_2 can be treated similar. One has to extract an (m, p, c) -set M out of $\cup_{x \in X_1} M_x$ such that $\Delta \upharpoonright M$ is one-to-one. Voigt [5] has developed general methods for such problems, but here a more simple way is applicable.

An (m, p, c) -set with the required properties will be built up recursively. Let w.l.o.g. $X_1 = \{0, \dots, m\}$. Put $y_m = x'_m$, so $M^{p,c}(y_m) = \{cy_m\}$ is a $(0, p, c)$ -set. Let an $(m-i, p, c)$ -set $M^{p,c}(y_i, \dots, y_m) = \cup_{j=i}^m M_j^{p,c}(y_j, \dots, y_m)$ with the following properties be constructed:

- $M_j^{p,c}(y_j, \dots, y_m) \subseteq M_j$ for every $j = i, i+1, \dots, m$;
- $M_j^{p,c}(y_j, \dots, y_m) \cap M_k^{p,c}(y_k, \dots, y_m) = \emptyset$ for every $j \neq k$ in $[i, m]$;
- $\Delta \upharpoonright M^{p,c}(y_i, \dots, y_m)$ is one-to-one.

Consider the set $M_{i-1} = \{cx'_{i-1} + \alpha x_i | \alpha \in [0, n_i]\}$. Since $|M^{p,c}(y_i, \dots, y_m)| \leq ((2p+1)^{m-i+1} - 1)/(1/2p)$ and $\Delta \upharpoonright M_{i-1}$ is one-to-one, there is $a_i \in [0, n_i]$ such that $\{cx'_{i-1} + a_i x_i + \alpha x_i | \alpha \in [0, 2A_i]\}$ is a subset of M_{i-1} and $\Delta(\{cx'_{i-1} + a_i x_i + \alpha x_i | \alpha \in [0, 2A_i]\}) \cap \Delta(M^{p,c}(y_i, \dots, y_m)) = \emptyset$. Put $y_{i-1} = (cx'_{i-1} + (a_i + A_i)x_i) \cdot c^{-1}$. Since A_i was chosen to be sufficiently large and $(*)$ is valid, $M_{i-1}^{p,c}(y_{i-1}, \dots, y_m) = \{cy_{i-1} + \sum_{j=i}^m \alpha_j y_j | \alpha_j \in [-p, p]\}$ is a subset of M_{i-1} and $M_{i-1}^{p,c}(y_{i-1}, \dots, y_m) \cup M^{p,c}(y_i, \dots, y_m)$ is an $(m-i+1, p, c)$ -set.

Finally obtain an (m, p, c) -set $M = \cup_{i=0}^m M_i^{p,c}(y_i, \dots, y_m)$ such that $\Delta \upharpoonright M$ is one-to-one and $M_i^{p,c}(y_i, \dots, y_m) \cap M_j^{p,c}(y_j, \dots, y_m) = \emptyset$ for every $i \neq j$ in $[0, m]$. ■

The following theorem provides a canonical version for homogeneous partition regular systems of linear equations:

2.2 THEOREM. *Let $A \in \mathbb{Z}^{v \times w}$ be a matrix with column property and let I_0, I_1, \dots, I_m be a corresponding partition of the set of column indices $[1, w]$ of A . Then there exists a natural number n which satisfies:*

for every coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ there is a solution $(\alpha_1, \dots, \alpha_w)^t \in [1, n]^w$ of $Ax = 0$ such that one of the following cases is valid:

- (i) $\Delta \upharpoonright \{\alpha_1, \dots, \alpha_w\}$ is constant;
- (ii) $\Delta \upharpoonright \{\alpha_1, \dots, \alpha_w\}$ is one-to-one;
- (iii) each two elements α_i, α_j of $\{\alpha_1, \dots, \alpha_w\}$ are colored the same with respect to Δ iff $\{i, j\} \subseteq I_k$ for one $k \in [0, m]$. (*)

The number of these cases is minimal for the class Π of homogeneous partition regular systems of linear equations, i.e., there is a system $Ax = 0$ in Π such that cases (i), (ii) do not suffice.

Proof. Obviously, (i) and (ii) are necessary, since the natural numbers can be colored constantly or injectively. But these do not suffice, as one easily verifies by considering the system of linear equations corresponding to a $(1, p, 1)$ -set and the coloring mentioned in the beginning of the proof of Theorem 2.1.

Now it will be shown that these cases are sufficient to describe the behaviour of the solutions.

Let $A \in \mathbb{Z}^{v \times w}$ be a matrix with column property and $[1, w] = I_0 \cup \dots \cup I_m$ be a corresponding partition of the set of column indices of A .

Let $c \in \mathbb{N}$ be the least common multiple of the denominators occurring in the linear combinations of the columns of A with respect to the partition I_0, \dots, I_m . Thus for every $j = 0, 1, \dots, m-1$ there exists integers s_{1j}, \dots, s_{vj} with

$$\sum_{i \in I_0 \cup \dots \cup I_j} a^i s_{ij} + \sum_{i \in I_{j+1}} ca^i = 0. \quad (*)$$

Let $p = \max\{c, |s_{ij}| \mid i \in [1, v], j \in [0, m-1]\}$ and let n be a natural number with $n \rightarrow (m, p, c)_{can}$.

Let $\Delta: [1, n] \rightarrow \mathbb{N}$ be a coloring. By Theorem 2.1 there is an (m, p, c) -set $M^{p,c}(x_0, \dots, x_m)$ in $[1, n]$ with $M_i^{p,c}(x_i, \dots, x_m) \cap M_j^{p,c}(x_j, \dots, x_m) = \emptyset$ for every $i \neq j$ in $[0, m]$ such that one of the following cases occurs:

- (i) $\Delta \upharpoonright M^{p,c}(x_0, \dots, x_m)$ is constant;
- (ii) $\Delta \upharpoonright M^{p,c}(x_0, \dots, x_m)$ is one-to-one;
- (iii) $\Delta \upharpoonright M_i^{p,c}(x_i, \dots, x_m)$ is constant for $i = 0, \dots, m$ and $\Delta \upharpoonright \{cx_0, \dots, cx_m\}$ is one-to-one.

As done in [1], construct recursively a solution of $A\mathbf{x} = \mathbf{0}$ which is contained in $M^{p,c}(x_0, \dots, x_m)$. Let $A^{(j)}$ be the submatrix of A which only consists of the columns \mathbf{a}^i of A with $i \in I_0 \cup \dots \cup I_j$. Obviously $cx_0 \sum_{i \in I_0} \mathbf{a}^i = \mathbf{0}$, so there is a solution of $A^{(0)}\mathbf{x} = \mathbf{0}$ in $M_0^{p,c}(x_0, \dots, x_m)$.

Let a solution of $A^{(j)}\mathbf{x} = \mathbf{0}$ be constructed, i.e.,

$$\sum_{i \in I_0 \cup \dots \cup I_j} \mathbf{a}^i y_i = \mathbf{0} \quad (**)$$

such that for every $i \in I_0 \cup \dots \cup I_j$ the following is valid: $i \in I_k$ implies $y_i \in M_k^{p,c}(x_k, \dots, x_m)$.

Multiplication of (*) by x_{j+1} and addition of (**) yields

$$\sum_{i \in I_0 \cup \dots \cup I_j} \mathbf{a}^i (y_i + s_{ij} x_{j+1}) + \sum_{i \in I_{j+1}} \mathbf{a}^i c x_{j+1} = \mathbf{0},$$

so a solution of $A^{(j+1)}\mathbf{x} = \mathbf{0}$ in $\bigcup_{k=0}^{j+1} M_k^{p,c}(x_k, \dots, x_m)$ is constructed.

Finally obtain a solution $(\alpha_1, \dots, \alpha_w)^t$ of $A\mathbf{x} = \mathbf{0}$ for which the following is valid for every $i \in [1, w]$:

$$i \in I_k \text{ implies } \alpha_i \in M_k^{p,c}(x_k, \dots, x_m).$$

By considering the coloring on $M^{p,c}(x_0, \dots, x_m)$ the theorem is proved. ■

Remark. By the proof of Theorem 2.1 it follows that for the constructed solution $(\alpha_1, \dots, \alpha_w)^t$ of $A\mathbf{x} = \mathbf{0}$ the following is valid for every $i, j \in [1, w]$:

α_i and α_j are different if i and j belong to different blocks of the partition I_0, \dots, I_m .

The other implication does not hold in general.

As a corollary we immediatly get a canonical version of the so-called Rado–Folkman–Sanders theorem on finite sums, which is a result of Prömel and Voigt.

2.3 COROLLARY [3]. *For every natural number m there exists a natural number n with the property:*

for every coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ there are mutually distinct positive integers x_0, \dots, x_m with $\sum_{i=0}^m x_i \leq n$ such that one of the following cases is valid for all nonempty subsets I, J of $[0, m]$:

- (i) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$;
- (ii) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$ iff $\min I = \min J$;
- (iii) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$ iff $I = J$.

The number of these cases is minimal.

Proof. By considering the coloring $\Delta: \mathbb{N} \rightarrow \mathbb{N}$ with $\Delta(x) = x' + 1$, where x has the representation $x = 2^{x'}x''$ with x'' odd, it is obvious that there are at least three cases. It remains to show that these suffice.

Let m be a natural number and put $M = 2^{2m+1}$. Consider the system of equations corresponding to the Rado–Folkman–Sanders theorem:

$$\sum_{i \in I} x_i = x_I, \quad I \subseteq [0, M], I \neq \emptyset$$

resp. $Ax = 0$.

For $i = 0, 1, \dots, M$ consider the columns of A belonging to x_i and x_I with $\min I = i$ and let I_i be the set of their indices. Obviously, I_0, \dots, I_M is a partition of the set of column indices of A and A has with respect to this partition column property.

For the matrix A and the partition I_0, \dots, I_M of the set of column indices let n be a natural number such that statement (*) of Theorem 2.2 is valid.

Let $\Delta: [1, n] \rightarrow \mathbb{N}$ be a coloring. By Theorem 2.2 and the subsequent remark there exists mutually distinct positive integers x_0, \dots, x_M with $\sum_{i=0}^M x_i \leq n$ such that one of the following cases is valid for all nonempty subsets I, J of $[0, M]$:

- (i) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$;
- (ii) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$ iff $\sum_{i \in I} x_i = \sum_{i \in J} x_i$;
- (iii) $\Delta(\sum_{i \in I} x_i) = \Delta(\sum_{i \in J} x_i)$ iff $\min I = \min J$.

By choice of M there exists an $(m+1)$ -element subset X of $\{0, 1, \dots, M\}$ such that for all nonempty subsets I, J of X it is valid: $\sum_{i \in I} x_i = \sum_{i \in J} x_i$ iff $I = J$; thus the corollary is proved. ■

3. INHOMOGENEOUS EQUATIONS

Let us consider the case of a single partition regular linear equation first.

3.1 THEOREM. *Let a_1, \dots, a_w, b be integers and let $\sum_{i=1}^w a_i x_i = b$ be partition regular.*

Then for every coloring $\Delta: \mathbb{N} \rightarrow \mathbb{N}$ there exist natural numbers β_1, \dots, β_w with $\sum_{i=1}^w a_i \beta_i = b$ such that $\Delta \upharpoonright \{\beta_1, \dots, \beta_w\}$ is either constant or one-to-one.

Proof. Let a_1, \dots, a_w, b be integers and let $\sum_{i=1}^w a_i x_i = b$ be partition regular. By Theorem 1.6 one of the following statements is valid:

- (i) there is $a \in \mathbb{N}$ such that $\sum_{i=1}^w a_i a = b$;
- (ii) there is $a \in \mathbb{Z} - \mathbb{N}$ such that $\sum_{i=1}^w a_i a = b$ and $\sum_{i=1}^w a_i x_i = 0$ is partition regular.

For case (i) there is nothing to prove, so suppose that $\sum_{i=1}^w a_i a = b$ for a nonpositive integer a and $\sum_{i=1}^w a_i x_i = 0$ is partition regular.

Let $\Delta: \mathbb{N} \rightarrow \mathbb{N}$ be a coloring. This induces a coloring $\Delta': \mathbb{N} \rightarrow \mathbb{N}$ which is defined by $\Delta'(x) = \Delta(a + |a - 1|x)$. As shown in Section 2 there exists natural numbers β_1, \dots, β_w with $\sum_{i=1}^w a_i \beta_i = 0$ such that $\Delta' \upharpoonright \{\beta_1, \dots, \beta_w\}$ is either constant or one-to-one.

Thus $\Delta \upharpoonright \{a + |a - 1|\beta_1, \dots, a + |a - 1|\beta_w\}$ is either constant or one-to-one and $\sum_{i=1}^w a_i (a + |a - 1|\beta_i) = b$. ■

By techniques similar to that used in the proof of Theorem 3.1 the following can be verified for inhomogeneous partition regular systems of linear equations.

3.2 THEOREM. *Let $A \in \mathbb{Z}^{v \times w}$ be a matrix with column property and let I_0, \dots, I_m be a corresponding partition of the set of column indices $[1, w]$ of A . Let $\mathbf{b} \in \mathbb{Z}^{v \times 1}$, where $A\mathbf{x} = \mathbf{b}$ is partition regular.*

Then there exists a positive integer n such that for every coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ there is a solution $(\beta_1, \dots, \beta_w)' \in [1, n]^w$ of $A\mathbf{x} = \mathbf{b}$ such that one of the following cases is valid:

- (i) $\Delta \upharpoonright \{\beta_1, \dots, \beta_w\}$ is constant;
- (ii) $\Delta \upharpoonright \{\beta_1, \dots, \beta_w\}$ is one-to-one;
- (iii) *each two elements β_i, β_j of $\{\beta_1, \dots, \beta_w\}$ are colored the same with respect to Δ iff $\{i, j\} \subseteq I_k$ for one $k \in [0, m]$.*

4. CONCLUDING REMARKS

For a coloring $\Delta: [1, n] \rightarrow \mathbb{N}$ the fibres $\Delta^{-1}(i)$ of the colors induce an equivalence relation on $[1, n]$: with respect to Δ two elements of $[1, n]$ are equivalent iff they are colored the same. Since one is only interested in the pattern of the coloring on a set, one can also use the notion of equivalence relation on a set.

Theorem 2.1 provide the following three equivalence relations on a 3-term arithmetic progression $a, a + d, a + 2d$ with its difference d , where $a \neq d$ in the sequel:

- (i) all in one class;
- (ii) each two different elements in different classes;
- (iii) the arithmetic progression in one class and the difference in another class.

Following Prömel and Voigt [3], a subset E' of the set E of all equivalence relations on $\{a, a + d, a + 2d, d\}$ is canonical iff for every color-

ing $\mathcal{A}: \mathbb{N} \rightarrow \mathbb{N}$ there exists a 3-term arithmetic progression $a, a+d, a+2d$ such that the equivalence relations on $\{a, a+d, a+2d, d\}$ induced by \mathcal{A} is contained in E' . Thus the subset E' of E described by (i), (ii), (iii) is a canonical set for a 3-term arithmetic progression with its difference. Moreover, E' is a (with respect to cardinality) minimal canonical set.

Obviously, there exist partitions of the natural numbers such that every 3-term arithmetic progression with its difference is of type (i) resp. (ii), but there is no partition such that every 3-term arithmetic progression with its difference is of type (iii). Thus every canonical set must contain the equivalence relations described by (i), (ii).

In [3] the question was raised whether all (with respect to set-inclusion) minimal canonical sets have the same cardinality. The answer is negative. 3-term arithmetic progressions with their difference provide an example for which minimal canonical sets (with respect to set-inclusion) have different cardinalities. Consider the colorings $\mathcal{A}_i: \mathbb{N} \rightarrow \mathbb{N}$, $i = 1, 2$, which are defined in the following way.

For every natural numbers x, y put

$$\mathcal{A}_1(x) = \mathcal{A}_1(y) \quad \text{iff } x' = y'$$

and

$$\mathcal{A}_2(x) = \mathcal{A}_2(y) \quad \text{iff } x' = y' \text{ and } x'' \equiv y'' \pmod{3}$$

where x, y have the representations $x = 3^{x'}x''$, $y = 3^{y'}y''$ with $x'', y'' \not\equiv 0 \pmod{3}$.

Obviously, the set E' of all equivalence relations except (iii) on $\{a, a+d, a+2d, d\}$ is a canonical set. The colorings $\mathcal{A}_1, \mathcal{A}_2$ show that every minimal canonical subset of E' has at least four elements. This shows that in general canonical sets are not uniquely determined; moreover, the minimal canonical sets (with respect to set-inclusion) do not have the same cardinality.

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